PHYS 1902 Assignment 2

Due: Monday, March 26, 2018
AGAIN, DON'T PANIC

Once again, this assignment's bark is worse than its bite. Please ask for help if you need it. **And please don't delay trying the assignment, term is ending faster than it seems!**

1. **The Galactic Mass Fraction of Interstellar Dust**: Interstellar gas and dust mix to form giant clouds and complexes in the galaxy. Dust affects the way wee see the galaxy because of general obscuration (interstellar extinction) and by selective blue light extinction (interstellar reddening); yet dust makes up a tiny fraction of the total mass in the galaxy. We can estimate the mass fraction of dust at our location in the galaxy using some basic inputs. 20 marks

   (a) Let's approximate dust grains as spheres, each with a radius of $R = 10^{-5}$ cm. Each grain of dust presents a cross-sectional shadow of πR^2. Recall that a photon's mean-free-path, l, is the typical distance a photon flies before bumping into matter. If the number density of dust grains (number of grains per cm^3) is n, the mean-free-path length relationship to the cross-sectional shadow is:

   $$l = \frac{1}{n\pi R^2} \tag{1}$$

   Given that interstellar extinction observations indicate that $l = 3 \times 10^3$ light-years, calculate the number density, n, of dust grains in units of cm^{-3}. How many dust grains would you expect to find in a volume equal to the Rogers Centre in Toronto (a cube about 100 metres in all directions)?

   (b) Assuming that dust grains have a typical matter density of about 2 gm/cm^3, estimate the mass, m_{grain}, of a dust grain. A volume $V = 300\,(\text{lty})^3$ in the galaxy typically contains one solar mass of stars. What is the mass of all the dust in this volume, $M_{\text{dust}} = nVm_{\text{grain}}$? What is the mass fraction of dust compared to stars at our position in the galaxy?

   The interstellar material with its interactions with embedded stars and dilute radiation fields produces extremely rarefied matter that is far from thermodynamic equilibrium. The birth and death of stars intimately link to the interstellar material as old stars fill space with heavy elements and new stars form when the gas and dust become dense and cold enough to gravitationally collapse. Gravity and the second law of thermodynamics shape the material between the stars. The enriched material forms a new generation of stars and planets—and new life who asks where it comes from. As we look out into the Universe from our small stage, the interstellar material reminds us that we are all made of star dust, fashioned by gravity and thermodynamics.

2. **White Dwarf Stars and the Chandrasekhar Mass Limit**: In the last assignment we discussed pressure as force per unit area. Let's revisit that idea from the microscopic perspective. Imagine a cloud of small pellets all moving in random directions in a box. Now imagine one of the walls of the box as pellets bounce against it. Over the course of time t, as the pellets move to the wall of area A, they sweep out a volume of volume $= Avt$, where v is the velocity of the pellets heading toward the wall. The pellets reflecting off the wall generates pressure. The force of the reflection comes from the change in momentum (mass times velocity $p = m_P v$, m_P is the pellet's mass) of each pellet. In time t, half of the pellets are moving towards the wall and the other half are moving away after reflection and the change in momentum is twice p since the velocity switches sign (the collision with the wall is perfectly elastic, the particles don't lose energy in the collision). If the number of particles per unity volume is n, then the pressure must be: 20 marks

   $P =$ [number of particles per unit volume]$\times$[volume swept out by particles heading towards and bouncing off the wall]$\times$[change in momentum per unit time]$\div$[the area of the wall] $= (n)(Avt/2)(2p/t)/A = nvp$.

   In three dimensions, we would adjust this result by a factor of 3, but we won't worry about that here. We have the formula for pressure in terms of microscopic quantitites: $P = nvp$, where p is the particle momentum, $p = mv$.

   Now imagine an electron gas with number density n_e. This means that on average, the distance between electrons is $\Delta x = (1/n_e)^{1/3}$. Let us suppose that the electron gas is so dense that it's *degenerate* meaning that the rules of quantum mechanics become important. In particular, the gas obeys Pauli's exclusion principle, which prevents two fermions (electrons are fermions) from occupying the same quantum state, and Heisenberg's uncertainty principle, which tells us that it is impossible to define the position and the momentum of a particle through their product to an accuracy better than Planck's constant: $(\Delta x)(\Delta p) > h$. For our degenerate electron gas, these principles mean that $p = h/x = hn_e^{1/3}$. If the electrons in the gas are moving much less than the speed of light, then $v = p/m_e$. These conditions are met inside a typical white dwarf star—the ions, which are the nuclei of the atoms that make up the white dwarf, are relatively stationary, but the electrons form a *degenerate* gas inside the solid body. Unlike a main sequence star, in which pressure generated from thermal effects prevents gravitational collapse, it's the quantum mechanical degeneracy pressure of the electron gas that supports a white dwarf.

(a) Based the relationship $P_e = n_e v p = n_e p^2/m_e$, and $p = h n_e^{1/3}$, show that the electron degeneracy pressure is $P_e = h^2 n_e^{5/3}/m_e$. In reality, this result is modified by a factor of 0.0485—not too bad for our rough estimate!

(b) Recall that the atomic number, Z, counts the number of protons in the nucleus of the atom and that the atomic weight, A, counts both the number of protons and the number of neutrons. For example carbon has atomic number 6, and atomic weight 12; oxygen has atomic number 8, and atomic weight 16. The white dwarf has overall charge neutrality meaning that the ions of atomic number Z that make up the star have the number density relationship $Zn_+ = n_e$, where n_+ is the ion number density. A white dwarf is typically composed of carbon-oxygen and so the ratio of the atomic number to the atomic weight, Z/A, is about 0.5. Since the proton and the neutron have similar mass, we will approximate the neutron mass with the proton mass, m_p. The proton is significantly more massive than the electron, so the density of the white dwarf is approximately $\rho = A m_p n_+$ and so $n_e = Z\rho/(A m_p)$. Show that we can write the electron degeneracy pressure as

$$P_e = \frac{h^2}{m_e}\left(\frac{Z}{A}\right)^{5/3}\frac{\rho^{5/3}}{m_p^{5/3}} \tag{2}$$

(c) Recall from the first assignment that the central pressure required to hold up a self-gravitating sphere of mass M and radius R is approximately $P_c = GM^2/R^4$. If we take the density to be $\rho = M/R^3$ in our expression for P_e, and if we set $P_e = P_c$, show that the mass-radius relationship of a white dwarf is:

$$R = \frac{h^2}{G m_e m_p^{5/3}}\left(\frac{Z}{A}\right)^{5/3} M^{-1/3} \tag{3}$$

The actual relationship is modified by a factor of 0.114—again, not bad for our rough calculation. Notice that the radius of a white dwarf shrinks as the cube of the mass. Unlike a main sequence star, the more massive a white dwarf becomes, the smaller its radius. Using the 0.114 correction factor, and setting $Z/A = 0.5$, calculate the radius of a white dwarf that has the mass of the Sun; $M_{\rm sun} = 1.99 \times 10^{33}$ gm, $G = 6.67 \times 10^{-8}$ gm^{-1} cm^3 s^{-2}, $h = 6.63 \times 10^{-27}$ erg s, $m_e = 9.11 \times 10^{-28}$ gm, $m_p = 1.67 \times 10^{-24}$ gm.

White dwarfs are like stellar embers, slowly cooling as they reach thermal equilibrium with the Universe. Since the degeneracy pressure holds up the white dwarf, it can cool without shrinking. Eventually, the white dwarf will end up with crystallized ions and a degenerate electron gas moving around inside the lattice structure.

(d) So far we have been dealing with a white dwarf with a degenerate electron gas in which the electrons are whizzing around at speeds much less than the speed of light. As we increase the mass of a white dwarf, the radius shrinks and eventually the degenerate electron gas becomes *relativistic*, that is, the electrons move around at near light speed, c. Under such extreme conditions, we need to modify our approach. The pressure for the relativistic degenerate electron gas is,

$$P_e = \frac{hc}{m_p^{4/3}} \left(\frac{Z}{A}\right)^{4/3} \rho^{4/3}. \tag{4}$$

If we again use $\rho = M/R^3$, and we set $P_e = P_c$, show that the radius R drops out of the calculation and we end up with the result,

$$M_* = \left(\frac{Z}{A}\right)^2 \left(\frac{hc}{Gm_p^2}\right)^{3/2} m_p. \tag{5}$$

In actuality M_* is modified by a factor of 0.2. This result is the Chandrasekhar mass limit of a white dwarf star, named for Subrahmanyan Chandrasekhar, the Nobel prize winning Indian-American physicist. The Chandrasekhar limit is 1.4 solar masses (you can verify the result numerically, if you'd like). Above the Chandrasekhar mass, even the quantum mechanical degeneracy pressure of the electron gas cannot stop the white dwarf from gravitationally collapsing. If the white dwarf has a mass greater than 1.4 solar masses, the white dwarf will "drive" the electrons into the protons to form a neutron star. If the neutron star exceeds about three solar masses, even the quantum mechanical properties of the neutron star cannot halt the gravitational collapse—the stellar remnant will collapse forever, forming a black hole—the topic of our next question.

3. **Hawking's Evaporating Black Holes, and the Scale of Quantum Gravity:** The escape velocity of a body is the minimum speed required to leave its gravitational clutch. The formula for the escape velocity is $v_{\text{esc}} = \sqrt{2GM/r}$, where r is the radius of the body, and M is its mass. 15 marks

(a) Suppose the gravitation field of a body with mass M is so strong that you would need to reach the speed of light, c, to escape. The radius of this object is called the *Schwarzschild radius*, named after Karl Schwarzschild, the physicist who provided the first exact solution to Einstein's field equations of general relativity—and he did it while serving in the German army on the Russian front during the first world war. Using the escape velocity formula above, show that $R_{Sch} = 2GM/c^2$. What is the Schwarzschild radius of a 3 solar mass black hole ($3M_{\mathrm{sun}} = 5.97 \times 10^{33}$ gm, $G = 6.67 \times 10^{-8}$ gm^{-1} cm^3 s^{-2}, $c = 3.00 \times 10^{10}$ cm/s)?

Think about our collapsing star with a mass larger than three solar masses. As we learned in the last question, quantum mechanical degeneracy pressure is powerless to stop the collapse. Once the radius of the collapsing star reaches the *Schwarzschild radius*, light itself cannot escape. But the star continues to collapse forever. In truth, the situation is a bit more complicated. Einstein's theory of General Relativity becomes important for such strong gravity fields and since spacetime becomes highly distorted, it doesn't make much sense to talk about a "radius" as such. Nevertheless, we can think about the point where an outwardly traveling photon could just barely escape—this is the *event horizon*. If we imagine tracing out these barely-escape points by going around the collapsing star, we will find that the circumference is $2\pi R_{Sch}$ and that the surface area is $4\pi R_{Sch}^2 = 16\pi G^2 M^2/c^4$. That is, we can interpret the *Schwarzschild radius* as the "radius" of the black hole. We will not consider rotation, as rotating black holes make the situation even more complicated.

(b) In part (a), notice that the surface area of the event horizon, $A = 16\pi G^2 M^2/c^4$, depends on the square of the mass. This observation lead Stephen Hawking to suppose that in any natural process, the surface area of the event horizon must always increase, or at best, stay the same. In 1972, Jacob Bekenstein noticed that the statement looks very much like a thermodynamic law—it looks a bit like the second law of thermodynamics except instead of entropy increasing, it's the surface area of the event horizon. Jacob Bekenstein made the bold suggestion that the surface area of the event horizon is a measure of the black hole's entropy. But if a black hole behaves like a thermodynamic body, what would it mean for the black hole to have temperature? Thermal bodies emit radiation, so how can a black hole have thermal properties if nothing can escape?

In quantum mechanics the vacuum is not a quiet place. Virtual particles of all types are constantly coming into "existence" and annihilating. Since the uncertainty principle states that energy differences and time differences have the

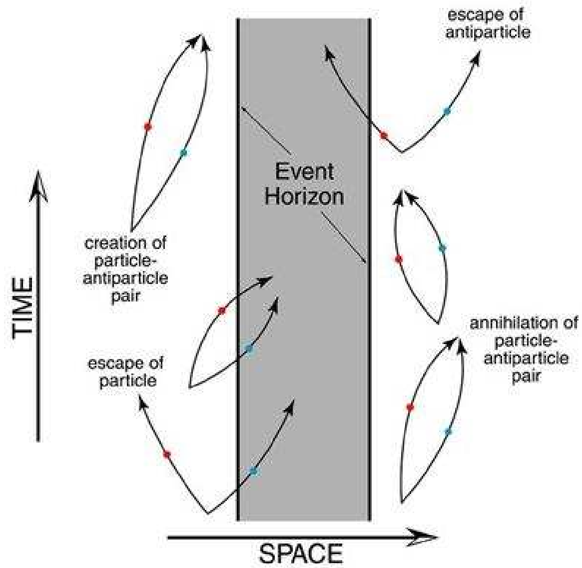

Figure 1: Hawking radiation near the event horizon of a black hole. The creation and annihilation of virtual particles usually ends with complete destruction unless the tidal forces cause one of the particles to fall into the black hole. In such cases, quantum mechanics allows the other particle to escape by "tunneling". The radiating particle reduces the mass of the black hole. Image Credit: Northern Arizona University

relationship $(\Delta E)(\Delta t) > h$, particle-antiparticle states can pop out of the vacuum with energy $\Delta E = mc^2$ as long as it returns the energy to the Universe inside the time $\Delta t \approx h/\Delta E$. We have lots of evidence for these vacuum processes in laboratory experiments. Stephen Hawking wondered about particle-antiparticle states emerging near the event horizon of a black hole. Normally, the particle-antiparticle state would quickly return the energy back to the Universe through annihilation, but suppose that one of the particles falls into the black hole before it gets a chance to annihilate with its partner. See figure 1. In that case, the event horizon will become a source of radiation. This particle emission from the event horizon, which carries away energy, steals mass from the black hole by the relationship $E = mc^2$. Stephen Hawking realized that black holes evaporate by thermal emission—a process we know today as Hawking radiation. Quantum mechanical processes[1] shrink the surface area of the event horizon and evaporate the black hole's mass!

Suppose that virtual particle-antiparticle "energy borrowing" from the uncertainty principle is governed by $\Delta E \Delta t = h/(4\pi)$. Let us further suppose that

[1]For stellar mass black holes, the evaporation process proceeds extremely slowly, taking many of orders of magnitude longer than the current age of the Universe for complete evaporation.

if the virtual pair separates by half the circumference of the event horizon, $c\Delta t/2 = 2\pi GM/c^2$, then one of the pair has a reasonable chance of falling into the black hole while the other one escapes. If the energy of the escaping particle has a thermal distribution, $\Delta E = kT$, where k is Boltzmann's constant, show that the temperature of the black hole is,

$$T = \frac{hc^3}{16\pi^2 kGM}. \tag{6}$$

This expression is Stephen Hawking's famous black hole radiation result. The event horizon is a source of radiation, and thus black holes evaporate with a temperature inversely proportional to the black hole's mass—small black holes are hotter than large ones. Compute the temperature of a 3 solar mass black hole, ($3M_{\text{sun}} = 5.97 \times 10^{33}$ gm, $G = 6.67 \times 10^{-8}$ gm^{-1} cm^3 s^{-2}, $h = 6.63 \times 10^{-27}$ erg s, $c = 3.00 \times 10^{10}$ cm/s, $k = 1.38 \times 10^{-16}$ erg/K).

Given the violent stages of the early universe, it is possible that micro black holes, with masses of a typical asteroid or terrestrial mountain (10^{16} gm), might have formed at that epoch. What would be the temperature of such a low mass black hole? These primordial micro black holes would have a Hawking radiation lifetime of about the current age of the Universe. Some astrophysicists have suggested that micro black holes could form the dark matter and we would be able to detect them from their Hawking radiation as they evaporate. The Fermi Gamma-ray Space Telescope is currently looking for gamma-ray burst signatures from evaporating primordial micro black holes.

(c) Imagine if we make the black hole quantum mechanically small so that twice its Compton wavelength, $2h/(mc)$, is equal to its Schwarzchild radius, $2Gm/c^2$. Show that $m = (hc/G)^{1/2}$. This mass, m_{Pl}, is called the Planck mass and on this scale gravity and quantum mechanics are both equally relevant. We don't know the laws of physics in this regime—it is the scale of quantum gravity. Compute the Planck mass in grams ($G = 6.67 \times 10^{-8}$ gm^{-1} cm^3 s^{-2}, $h = 6.63 \times 10^{-27}$ erg s, $c = 3.00 \times 10^{10}$ cm/s) and compare it to the mass of the recently discovered Higgs Boson, $M_{\text{Higgs}} = 2.2 \times 10^{-22}$ gm. What does the scale difference tell you about how much more powerful our particle accelerators need to become before we can directly explore this physics?

In this course, we have seen how astrophysical processes involve the competition between gravitation and the second law of thermodynamics. A black hole is the most pure gravitational stellar end state, but as we have seen from Hawking radiation, quantum mechanics implies that even black holes are thermodynamic bodies. What is the ultimate end state for matter? If grand unification ideas in particle physics are correct, all atomic nuclei are unstable, eventually decaying into photons and leptons. If those ideas are correct, even white dwarfs and neutron stars will decay away. The observation that the Universe's expansion is accelerating suggests that the second law of thermodynamics will win out. The Universe will expand forever, eventually reaching thermodynamic equilibrium with no black holes, no stars, no galaxies, no planets—all material entities being just way-stations on Nature's journey of turning all matter into photons and leptons.

55 marks

"Vacuums, black holes, antimatter - it's the elusive and intangible which appeals to me."